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LETTER TO THE EDITOR

Affine Hecke algebra, Macdonald polynomials, and quantum many-body systems

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Abstract. The Macdonald operators associated with the classical root systems are constructed based on the infinite-dimensional representation for solutions of the Yang–Baxter equation and the reflection equation.

There has been renewed interest in the q-deformed orthogonal polynomials related with quantum groups. One of the famous orthogonal polynomials is the Macdonald polynomial associated with root systems [1]. Some polynomials, such as the Rogers–Askey–Ismail polynomial and the Askey–Wilson polynomial [2], can be regarded as special cases of the Macdonald polynomial. The A_{N-1}-type Macdonald polynomial is an eigenfunction of

$$\hat{M}_n^{\mathrm{Macdo}} = \sum_{\substack{I\subset\{1,2,\ldots,N\}\\ |I|=n}} \prod_{\substack{j\in I\\ k\notin I}} \frac{q^{-1}z_j - qz_k}{z_j - z_k} \prod_{j\in I} \hat{T}_j \tag{1}$$

where the shift operator is defined as

$$(\hat{T}_j f)(\ldots, z_j, \ldots) = f(\ldots, pz_j, \ldots).$$

On the other hand the BC_{N-1}-type Macdonald operator (or, Macdonald–Koornwinder operator) is defined as the difference operator

$$\hat{M}_1^{\mathrm{BC}} = \sum_{j=1}^{N} \Psi_j(z)\cdot(\hat{T}_j - 1) + \sum_{j=1}^{N} \Psi_j(z^{-1})\cdot(\hat{T}_j^{-1} - 1) \tag{2}$$

where we set

$$\Psi_j(z) = \frac{(1-az_j)(1-bz_j)(1-cz_j)(1-dz_j)}{(1-z_j^2)(1-pz_j^2)} \cdot \prod_{\substack{k=1\\ k\neq j}}^{N} \frac{(tz_j - z_k)(1-tz_jz_k)}{(z_j-z_k)(1-z_jz_k)}.$$

Note that we have five arbitrary parameters, $\{a, b, c, d, p\}$. The polynomial as eigenfunctions for this difference operator is introduced as a generalization of the Askey–Wilson polynomial.

Recently the relationship between the Macdonald polynomial and the affine Hecke algebra has been revealed [3, 4]. The Macdonald operators are constructed as the quantum

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Knizhnik–Zamolodchikov type operator. In this letter, the Macdonald operators are studied based on solutions of the Yang–Baxter equation and the reflection equation. We give the infinite-dimensional representation for solutions, and construct the integrable difference operator associated with the classical root systems.

Let us consider solutions of the Yang–Baxter equation (YBE) and the reflection equation (RE, or boundary Yang–Baxter equation), which are respectively written as

$$R^{12}(\theta_1/\theta_2)R^{13}(\theta_1/\theta_3)R^{23}(\theta_2/\theta_3) = R^{23}(\theta_2/\theta_3)R^{13}(\theta_1/\theta_3)R^{12}(\theta_1/\theta_2) \tag{3}$$

$$R^{12}(\theta_1/\theta_2)(K(\theta_1)\otimes 1)R^{21}(\theta_1\theta_2)(1\otimes K(\theta_2)) \\ = (1\otimes K(\theta_2))R^{12}(\theta_1\theta_2)(K(\theta_1)\otimes 1)R^{21}(\theta_1/\theta_2). \tag{4}$$

Here θ_j are called the spectral parameters. The YBE is an elementary tool to investigate integrable systems (see, [5–7]). The RE is used to formulate integrable systems with boundary [8]. As a solution of the YBE and RE, we use the operator-valued solution as the 'infinite-dimensional' representation [9–13]; both R and K are regarded as operators acting on the functional space $f(z_1,\ldots,z_N)$. Although elliptic solutions are explicitly given, we only study the trigonometric solutions as degenerate cases. The R-operator is defined by

$$R^{jk}(\theta) = \frac{-1}{q-q^{-1}\theta}(\theta\hat{g}_{j,k} - \hat{g}_{j,k}^{-1})\cdot\hat{s}_{j,k} \tag{5}$$

where q is an arbitrary parameter, and $\hat{g}$ is the Demazure–Lusztig (DL) operator

$$\hat{g}_{j,k} = \frac{q^{-1}z_j - qz_k}{z_j - z_k}\hat{s}_{j,k} + (q-q^{-1})\frac{z_k}{z_j - z_k}. \tag{6}$$

It is noted that the operator $\hat{s}_{j,k}$ exchanges coordinates

$$(\hat{s}_{j,k}f)(\ldots,z_j,\ldots,z_k,\ldots) = f(\ldots,z_k,\ldots,z_j,\ldots).$$

As solutions for the RE (4) associated with the above trigonometric R-operator (5), there are two solutions, K and $\bar{K}$ [12]:

$$K^j(\theta) = \frac{1}{a-a^{-1}\theta^2}(\theta^2\hat{r}_j - \hat{r}_j^{-1}) \tag{7a}$$

$$\bar{K}^j(\theta) = \frac{1}{b-b^{-1}\theta}(\theta\hat{\bar{r}}_j - \hat{\bar{r}}_j^{-1}) \tag{7b}$$

where parameters a and b are arbitrary, and we define the DL operators $\hat{r}$ as

$$\hat{r}_j = \frac{a-a^{-1}}{z_j - 1} + \frac{a^{-1}z_j - a}{z_j - 1}\hat{t}_j \tag{8a}$$

$$\hat{\bar{r}}_j = \frac{b-b^{-1}}{z_j^2 - 1} + \frac{b^{-1}z_j^2 - b}{z_j^2 - 1}\hat{t}_j. \tag{8b}$$

Operator $\hat{t}$ is a reflection operator, which acts on a functional space as

$$(\hat{t}_j f)(\ldots,z_j,\ldots) = f(\ldots,z_j^{-1},\ldots).$$

These two solutions correspond to a reflection at the boundary associated with the Weyl group of type B and type C, respectively.

We note several identities for the DL operators, which may be directly checked:

$$\hat{g}_{j,j+1}\hat{g}_{j+1,j+2}\hat{g}_{j,j+1} = \hat{g}_{j+1,j+2}\hat{g}_{j,j+1}\hat{g}_{j+1,j+2} \tag{9a}$$

$$(\hat{g}+q)(\hat{g}-q^{-1}) = 0 \tag{9b}$$

$$\hat{r}_j\hat{g}_{j,j+1}\hat{r}_j\hat{g}_{j,j+1} = \hat{g}_{j,j+1}\hat{r}_j\hat{g}_{j,j+1}\hat{r}_j \tag{9c}$$

$$(\hat{r}-a)(\hat{r}+a^{-1}) = 0. \tag{9d}$$

Both operators $\hat{g}$ and $\hat{r}$ satisfy the Hecke relations. We have the same identities for operator $\hat{\bar{r}}$. Using the DL operators, we can define sets of integrable difference operators, which have the same structure with the quantum Knizhnik–Zamolodchikov (qKZ) operators [14]:

$$\hat{D}_j = \hat{g}_{j,j-1}\hat{g}_{j-1,j-2}\cdots\hat{g}_{2,1}\cdot\hat{T}_1\cdot\hat{g}^{-1}_{1,N}\hat{g}^{-1}_{N,N-1}\hat{g}^{-1}_{N-1,N-2}\cdots\hat{g}^{-1}_{j+2,j+1}\hat{s}_{1,N}\hat{s}_{1,N-1}\cdots\hat{s}_{1,2} \tag{10}$$

$$\hat{Y}_j = \hat{g}_{j,j-1}\hat{g}_{j-1,j-2}\cdots\hat{g}_{2,1}\cdot\hat{T}_1\hat{r}_1\hat{T}_1^{-1}\cdot\hat{g}_{2,1}\hat{g}_{3,2}\cdots\hat{g}_{N,N-1}\hat{\bar{r}}_N^{-1}\cdot\hat{g}^{-1}_{N,N-1}\hat{g}^{-1}_{N-1,N-2}\cdots\hat{g}^{-1}_{j+1,j}. \tag{11}$$

We can see from the YBE (3) and RE (4) that they constitute commuting families,

$$[\hat{D}_j, \hat{D}_k] = 0 \tag{12}$$

$$[\hat{Y}_j, \hat{Y}_k] = 0 \qquad \text{for } \forall_{j,k} = 1, \ldots, N. \tag{13}$$

The qKZ operators are shown to satisfy the following relations:

$$\hat{g}_{j+1,j}\hat{D}_j\hat{g}_{j+1,j} = \hat{D}_{j+1}$$
$$\hat{g}_{j+1,j}\hat{Y}_j\hat{g}_{j+1,j} = \hat{Y}_{j+1}$$
$$\hat{Y}_1^{-1}\cdot(\hat{T}_1\hat{r}_1\hat{T}_1^{-1}) = (\hat{T}_1\hat{r}_1\hat{T}_1^{-1})^{-1}\cdot\hat{Y}_1 - (b - b^{-1}).$$

We remark that algebra constructed from operators $\{\hat{D}_j, \hat{g}_{j+1,j} | j = 1, \ldots, N\}$ is called as the degenerate affine Hecke algebra of type A while algebra from $\{\hat{Y}_j, \hat{g}_{j+1,j}, \hat{T}_1\hat{r}_1\hat{T}_1^{-1}, \hat{\bar{r}}_N\}$ as algebra of type BC. Commuting difference operators give us sets of integrable difference operators:

$$\hat{M}_n = \sum_{j=1}^{N}(\hat{D}_j)^n \tag{14}$$

$$\hat{W}_n = \sum_{j=1}^{N}((\hat{Y}_j)^n + (\hat{Y}_j)^{-n}). \tag{15}$$

In the rest of this letter we shall clarify that two sets of operators $\{\hat{D}_j | j = 1, \ldots, N\}$ and $\{\hat{Y}_j | j = 1, \ldots, N\}$ gives the A_{N-1}- and BC_{N-1}-type Macdonald operators, respectively. We also show that they reduce in the quasi-classical limit to the Dunkl operator for the N-body Hamiltonian of the Calogero–Sutherland–Moser (CSM) model, which is a one-dimensional integrable system with inverse-square interactions [15].

We study the simple case in more detail. Assuming that operators $\hat{D}_j$ (10) act on a symmetric functional space,

$$(\hat{s}_{j,k}f)(z) = f(z) \qquad \text{for } {}^\forall j, k \in \{1, 2, \ldots, N\}$$

we get the difference operator for the $N = 2$ case as

$$\hat{M}_1 = \frac{q^{-1}z_1 - qz_2}{z_1 - z_2}\hat{T}_1 + \frac{q^{-1}z_2 - qz_1}{z_2 - z_1}\hat{T}_2.$$

This is nothing but the Macdonald operator for type A_1 (1). Eigenfunctions of the A_1-type Macdonald operator are explicitly given from the generating function $F(z; t)$ [1],

$$F(z; t) = \frac{(q^{-2}z_1t; p)_\infty}{(z_1t; p)_\infty}\cdot\frac{(q^{-2}z_2t; p)_\infty}{(z_2t; p)_\infty} \tag{16}$$

where the q-product is defined as

$$(a; q)_n = (1 - a)(1 - aq)\cdots(1 - aq^{n-1}) \qquad (a; q)_0 = 1.$$

Simple calculation results in the difference equation for function $F(z;t)$,

$$(\hat{M}_1 F)(z;t) = qF(z;t) + q^{-1}F(z;pt).$$

One can define the Rogers–Askey–Ismail (RAI) polynomial (continuous q-ultraspherical polynomial) $C_n(z)$ from the generating function $F(z;t)$ as

$$F(z;t) = \sum_{n=0}^{\infty} \frac{(q^{-2};p)_n}{(p;p)_n} C_n(z) t^n. \tag{17}$$

The difference equation for $F(z;t)$ indicates that the RAI polynomial becomes an eigenfunction of the Macdonald operator,

$$\hat{M}_1 C_n(z) = (q + q^{-1}p^n) C_n(z). \tag{18}$$

One concludes that the RAI polynomial $C_n(z)$ coincides with the A_1-Macdonald polynomial $P_{[n]}(z)$ [16]. The polynomials for the arbitrary Young diagram $\lambda = [\lambda_1, \lambda_2]$ ($\lambda_1 \geqslant \lambda_2$) are calculated from an identity [1],

$$P_{[\lambda_1,\lambda_2]}(z) = (z_1 z_2)^{\lambda_2} \cdot C_{\lambda_1-\lambda_2}(z).$$

It should be remarked that in the limit $q^{-2} \to 0$ the RAI polynomial reduces to the Rogers–Szegö polynomial, whose recurrence relation is shown to give the representation for the Yangian invariant bases called 'motif' [17].

The difference operator $\hat{W}_1$ (15) is calculated by restricting the functional space to symmetric space,

$$(\hat{s}_{j,k} f)(z) = f(z) \qquad (\hat{t}_j f)(z) = f(z) \qquad \text{for } {}^{\forall} j, k \in \{1, 2, \ldots, N\}.$$

We obtain the difference operator for $N = 2$ as

$$\begin{aligned}\hat{W}_1 = {} & (ab^{-1}q + a^{-1}bq^{-1})(q + q^{-1}) + \Phi(z_1, z_2) \cdot (\hat{T}_1^2 - 1) + \Phi(z_2, z_1) \cdot (\hat{T}_2^2 - 1) \\ & + \Phi(z_1^{-1}, z_2) \cdot (\hat{T}_1^{-2} - 1) + \Phi(z_2^{-1}, z_1) \cdot (\hat{T}_1^{-2} - 1)\end{aligned}$$

where function $\Phi(x, y)$ means

$$\Phi(x, y) = \frac{q^{-1}x - qy}{x - y} \cdot \frac{a^{-1}p^2x^2 - a}{p^2x^2 - 1} \cdot \frac{bx - b^{-1}}{x - 1} \cdot \frac{q^{-1}xy - q}{xy - 1}.$$

This operator is for the Askey–Wilson polynomial (2) with three arbitrary parameters [18, 19].

We can calculate the difference operators $\hat{M}_1$ and $\hat{W}_1$ for arbitrary N in the same manner. The Macdonald operator $\hat{M}_1$ (14) is calculated by restricting to the symmetric functional space as

$$\hat{M}_1 = \sum_{j=1}^{N} \left(\prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{q^{-1}z_j - qz_k}{z_j - z_k} \right) \hat{T}_j. \tag{19}$$

In general, it is not possible to give all eigenfunctions explicitly for arbitrary Young diagrams. Some of eigenfunctions are simply given as the 'generalized' RAI polynomial. We introduce the generating function $F^{(N)}(z;t)$ and the generalized RAI polynomials $C_n^{(N)}(z)$ as

$$F^{(N)}(z;t) = \prod_{j=1}^{N} \frac{(q^{-2}z_j t;p)_\infty}{(z_j t;p)_\infty} = \sum_{n=0}^{\infty} \frac{(q^{-2};p)_n}{(p;p)_n} C_n^{(N)}(z) t^n. \tag{20}$$

One finds that the function $F^{(N)}(z;t)$ satisfies the difference equation,

$$(\hat{M}_1 F^{(N)})(z;t) = \left(\sum_{j=1}^{N-1} q^{2j-N+1}\right) F^{(N)}(z;t) + q^{-N+1} F^{(N)}(z;pt)$$

which proves that the generalized RAI polynomial is an eigenfunction of $\hat{M}_1$,

$$(\hat{M}_1 C_n^{(N)})(z) = \left(\sum_{j=1}^{N-1} q^{2j-N+1} + q^{-N+1} p^n\right) C_n^{(N)}(z). \tag{21}$$

As in the case for $N = 2$, the generalized RAI polynomial $C_n^{(N)}(z)$ is indeed the A_{N-1}-Macdonald polynomial $P_\lambda(z)$ for Young diagram $\lambda = [n]$. The polynomials for other λ can be given by using the orthogonality of the Macdonald polynomials recursively [1].

The integrable difference operators for BC-type are calculated as follows,

$$\hat{W}_1 = \Phi_0 + \sum_{j=1}^{N} \Phi_j(z) \cdot (\hat{T}_j^2 - 1) + \sum_{j=1}^{N} \Phi_j(z^{-1}) \cdot (\hat{T}_j^{-2} - 1) \tag{22}$$

where functions $\Phi_j(z)$ and Φ_0 are defined by

$$\Phi_j(z) = \left(\prod_{\substack{k=1\\k\neq j}}^{N} \frac{q^{-1}z_j - qz_k}{z_j - z_k} \cdot \frac{q^{-1}z_j z_k - q}{z_j z_k - 1}\right) \cdot \frac{a^{-1}p^2 z_j^2 - a}{p^2 z_j^2 - 1} \cdot \frac{bz_j - b^{-1}}{z_j - 1}$$

$$\Phi_0 = (a^{-1}bq^{1-N} + ab^{-1}q^{N-1}) \sum_{k=1}^{N} q^{2k-N-1}.$$

This difference operator is indeed for the Macdonald–Koornwinder polynomial (2). As in the case of $N = 2$ we have only three parameters, not five.

Both the difference operators $\hat{D}_j$ and $\hat{Y}_j$ are regarded as the quantum Dunkl operators [13, 20]. When we take a quasi-classical limit in operators, we obtain a well known Dunkl operator associated with the classical root systems. For the A-type operator $\hat{D}_j$ (10), we set the parameter as

$$q = p^{-\beta}$$

and take a quasi-classical limit,

$$p \to 1 + \varepsilon + \mathrm{O}(\varepsilon^2).$$

After expanding the difference operator $\hat{D}_j$ in ε, we obtain a new set of operators,

$$\hat{D}_j = 1 + \varepsilon \hat{d}_j + \mathrm{O}(\varepsilon^2)$$

where $\hat{d}_j$ is the differential-difference operator, called the A-type (trigonometric) Dunkl operator, defined by

$$\hat{d}_j = z_j \frac{\partial}{\partial z_j} - 2\beta \sum_{k<j} \frac{z_k}{z_j - z_k}(\hat{s}_{j,k} - 1) - 2\beta \sum_{k>j} \frac{z_j}{z_j - z_k}(\hat{s}_{j,k} - 1) + \beta(2j - N - 1). \tag{23}$$

One sees from the commutativity of $\hat{D}_j$ that the Dunkl operator $\hat{d}_j$ is integrable,

$$[\hat{d}_j, \hat{d}_k] = 0 \qquad \text{for } {}^\forall j, k. \tag{24}$$

In the same manner, one obtains the BC-type Dunkl operator by taking the quasi-classical limit of the difference operator $\hat{Y}_j$ (11). We set, in this case, the parameters as

$$q = p^{-\beta} \qquad a = p^{-2\tilde{\alpha}} \qquad b = p^{2\alpha}.$$

One sees that the qKZ-type difference operator $\hat{Y}_j$ is expanded as

$$\hat{Y}_j = 1 + 2\varepsilon\hat{y}_j + \mathrm{O}(\varepsilon^2),$$

where the BC-type Dunkl operator $\hat{y}_j$ is given by

$$\begin{aligned}\hat{y}_j = z_j\frac{\partial}{\partial z_j} - \beta\sum_{k<j}\frac{z_k}{z_j - z_k}(\hat{s}_{j,k} - 1) - \beta\sum_{k>j}\frac{z_j}{z_j - z_k}(\hat{s}_{j,k} - 1)\\ -\beta\sum_{k\neq j}\frac{1}{z_j z_k - 1}(\hat{t}_j\hat{t}_k\hat{s}_{j,k} - 1) + \beta(j-1)\\ -\alpha\left(\frac{2}{z_j - 1}\hat{t}_j - \frac{z_j + 1}{z_j - 1}\right) - \bar{\alpha}\left(\frac{2}{z_j^2 - 1}\hat{t}_j - \frac{z_j^2 + 1}{z_j^2 - 1}\right).\end{aligned} \tag{25}$$

It is noted that the BC-type operator also constitutes an integrable family,

$$[\hat{y}_j, \hat{y}_k] = 0 \qquad \text{for } {}^\forall j, k. \tag{26}$$

The commutativity of the Dunkl operators $\hat{d}_j$ and $\hat{y}_j$ shows that sets of integrable Hamiltonians can be defined. In fact the Hamiltonians of the N-body quantum Calogero–Sutherland–Moser (CSM) model of type A [21] and type BC [22, 23] are explicitly given from the Dunkl operator as [13]

$$\tilde{\mathcal{H}}^{\mathrm{A}} = \sum_{j=1}^{N}\pi(\hat{d}_j^2) \qquad \tilde{\mathcal{H}}^{\mathrm{BC}} = \sum_{j=1}^{N}\pi(\hat{y}_j^2)$$

where we denote π as a restriction of the functional spaces to the symmetric case. One sees after lengthy calculation that the Hamiltonians of the CSM model are given by factorizing out products of functions,

$$\mathcal{H}^{\mathrm{A}} = \Delta^{\mathrm{A}}(z)\cdot\tilde{\mathcal{H}}^{\mathrm{A}}\cdot\Delta^{\mathrm{A}}(z)^{-1} = \sum_{j=1}^{N}\left(z_j\frac{\partial}{\partial z_j}\right)^2 - 4\beta(2\beta - 1)\sum_{1\leqslant j<k\leqslant N}\frac{z_j z_k}{(z_j - z_k)^2} \tag{27}$$

$$\begin{aligned}\mathcal{H}^{\mathrm{BC}} &= \Delta^{\mathrm{BC}}(z)\cdot\tilde{\mathcal{H}}^{\mathrm{BC}}\cdot\Delta^{\mathrm{BC}}(z)^{-1}\\ &= \sum_{j=1}^{N}\left(z_j\frac{\partial}{\partial z_j}\right)^2 - 2\beta(\beta - 1)\sum_{1\leqslant j<k\leqslant N}\left(\frac{z_j z_k}{(z_j - z_k)^2} + \frac{z_j z_k}{(z_j z_k - 1)^2}\right)\\ &\quad - \sum_{j=1}^{N}\left(\alpha(2\bar{\alpha} + \alpha - 1)\frac{z_j}{(z_j - 1)^2} + 4\bar{\alpha}(\bar{\alpha} - 1)\frac{z_j^2}{(z_j^2 - 1)^2}\right)\end{aligned} \tag{28}$$

where $\Delta^{\mathrm{A,BC}}(z)$ are the ground-state eigenfunctions for the A-type and BC-type CSM model, respectively,

$$\Delta^{\mathrm{A}}(z) = \prod_{j=1}^{N} z_j^{-\beta(N-1)}\cdot\prod_{1\leqslant j<k\leqslant N}(z_j - z_k)^{2\beta}$$

$$\Delta^{\mathrm{BC}}(z) = \prod_{j=1}^{N} z_j^{-\beta(N-1)-\bar{\alpha}-\alpha}(z_j - 1)^{2\alpha}(z_j^2 - 1)^{\bar{\alpha}}\cdot\prod_{1\leqslant j<k\leqslant N}(z_j - z_k)^{\beta}(z_j z_k - 1)^{\beta}.$$

In this sense the Macdonald operators $\hat{M}_1$ and $\hat{W}_1$ may be considered as the Hamiltonians for the 'relativistic CSM model' (the difference analogue of the CSM model) [24].

We have clarified the role of the affine Hecke algebra in the Macdonald polynomials and the Calogero–Sutherland–Moser models, both of which are associated with the root system.

Using the infinite-dimensional representation for solutions of the YBE and the RE, we have constructed a degenerate affine Hecke algebra of type A and type BC. Although our operators could be viewed as the Macdonald operators associated with the root system of type A and type BC, BC-type operators include only three parameters; the Macdonald–Koornwinder operator includes five arbitrary parameters in general. In [4] the Askey–Wilson polynomial with five parameters is constructed by use of the affine Hecke algebra. An $\hat{r}$-operator is used, which unifies our two solutions (8) and satisfies identities (9),

$$\hat{r}_j^{\text{Noumi}} = \frac{(a - a^{-1}) + (b - b^{-1})z_j}{1 - z_j^2} + \frac{a^{-1} - (b - b^{-1})z_j - az_j^2}{1 - z_j^2}\hat{t}. \tag{29}$$

It might be possible to unify two elliptic solutions of the reflection equation [12], and to construct the elliptic analogue of the Macdonald–Koornwinder operator [25].

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